ISRAEL JOURNAL OF MATHEMATICS, Vol. 22, Nos. 3-4, 1975

GOLDSCHMIDT'S 2-SIGNALIZER FUNCTOR THEOREM

BY

HELMUT BENDER[†]

ABSTRACT

A solvable A-signalizer functor θ assigns to any non-identity element x of the abelian 2-subgroup A of the finite group G an A-invariant solvable $2'$-subgroup $\theta(C_G(x))$ of $C_G(x)$ such that $\theta(C_G(x)) \cap C_G(y) \subseteq \theta(C_G(y))$ for all $x, y \in A^\#$. θ is called complete if G has a solvable A-invariant $2'$-subgroup $K = \theta(G)$ such that $C_K(x) = \theta(C_G(x))$ for every $x \in A^\#$.
This note contains an alternate proof of the completeness theorem below.

1. Introduction

The fundamental concept of a signalizer functor is due to Gorenstein. His results, see [5], [6], [7], [8], precede the following

THEOREM OF GOLDSCHMIDT ([3]). *Let G be a (finite!) group and A an abelian 2-subgroup of rank $r(A) \geqq 3$.*
Then any solvable A-signalizer functor θ on G is complete.

We show that the "maximal subgroup approach" underlying some other work of the author also yields a proof of this theorem, thereby answering a question raised in [8, p. 108] affirmatively.

The general signalizer functor theorem (replace 2 by an arbitrary prime r) has recently been proved by Glauberman [1]. Before, Goldschmidt [2] and Martineau [9] had done the cases $r(A) \geqq 4$ and $\theta(C_G(A)) = 1$, respectively.

For notation and a general discussion of signalizer functors the reader is refered to [2].

Here are the general facts we need, up to the ZJ-theorem all of a very elementary nature:

[†]This research originated in September 1973 while the author was visiting the University of Tel Aviv on a grant from the Deutsche Forschungsgemeinschaft.

Received February 23, 1975

1.1. *Standard action properties of a p-group P acting on a p'-group K*:

(i) *$[P,K]$ is a P-invariant normal subgroup of K satisfying $K = [P,K]C_K(P)$ and $[P,K]=[P,[P,K]]$*;

(i') *if P centralizes a centralizer-closed (i.e. $C_K(U)\subseteq U$) normal (or just subnormal) subgroup U of K, then P centralizes K*;

(i'') *if $X^P = X \lhd K = XY$ with $Y = Y^P$, then $C_K(P)=C_X(P)C_Y(P)$*;

(ii) *if P is abelian, then $K = \langle C_K(P_0) \mid P/P_0 \text{ cyclic}\rangle$*;

(ii') *if $T\subseteq P$, P abelian, then $[T,K]=\langle [T, C_K(P_0)] \mid P/P_0 \text{ cyclic}\rangle$*;

(iii) *any P-invariant q-subgroup (q a prime) of K lies in a P-invariant S_q-subgroup of K, and all such S_q-subgroups are conjugate under $C_K(P)$; in case K is solvable, this holds for any set q of primes*;

(iv) *if $|P|=2$ and $C_K(P)=1$, then K is abelian and $k^u = k^{-1}$ for $u\in P^*$ and every $k\in K$.*

1.2. *Standard properties of a solvable group H, with $\pi = \pi(F(H))$*:

(i) $C_H(F(H))\subseteq F(H)$;

(ii) $O_{p'}(C_H(P))\subseteq O_{p'}(H)$ *for any p-subgroup P*;

(iii) *if $C_{F(H)}(U)\subseteq U\subseteq F(H)$, then $C_H(U)$ is a (nilpotent) π-group.*

LEMMA 1.3. *Let t be an involution acting on the solvable 2'-group H. Let $C_H(t)\subseteq K = K'\subseteq H$. Then $[t,F(K)]\subseteq F(H)$.*

For the next result, basic for us, it is convenient to have some additional notation: For solvable subgroups M and K of a group G write $M*K$, $M**K$, or $M***K$, according as $(*)$, $(**)$, or both holds.

$(*)$ *$N_K(X)\subseteq M$ whenever $1\neq X$* char *M*,

$(**)$ *K contains a centralizer-closed subgroup of $F(M)$.*

PROPOSITION 1.4. *Assume $M***K$. Let $\pi = \pi(F(M))$ and $U = F(M)\cap K$.*

(i) $O_{\pi'}(K)\cap M = 1$;

(ii) $U\subseteq O_{\pi',F}(K) := F(K \bmod O_{\pi'}(K))$ *unless $|\pi|=1$*;

(iii) *if we also have $K***M$, then $M=K$ unless $F(M)$ and $F(K)$ are p-groups for some prime p.*

GLAUBERMAN ZJ-THEOREM ([4], theor. 8.2.11). *Let H be solvable, $F(H)$ a p-group, $p\geqq 5$ or $2\notin\pi(H)$, and $H_p\in \mathrm{Syl}_p(H)$. Then $Z(J(H_p))\lhd H$.*

In the following, θ is an A-signalizer functor on G, with $r(A)\geqq 3$. Subgroups in $\mathrm{II}_\theta(A)$ we also call θ-subgroups. By definition of $\mathrm{II}_\theta(A)$, A-invariant subgroups of θ-subgroups are θ-subgroups.

THOMPSON TRANSITIVITY THEOREM ([2], theor. 3.1). *For any prime* q, $\theta(C_G(A))$ *is transitive on* $\mathcal{M}^*_\theta(A,q)$.

1.5. (A corollary of 1.1.i″). *If* $X, Y \in \mathcal{M}_\theta(A)$, $X^Y = X$, *then* $XY \in \mathcal{M}_\theta(A)$.

1.6. (Glauberman [1]). *Let* θ *be solvable, and* π *a set of primes. Then, for any* $x \in A^*$, $K := \theta(C_G(x))$ *has a unique maximal* $\theta(C_G(A))$*-invariant (i.e.* $C_K(A)$*-invariant)* π*-subgroup* $\theta_\pi(C_G(x))$, *and the function* θ_π *thus defined is an* A*-signalizer functor on* G.

1.7. *Let* $\bar{G} = G/X$ *with* X *a normal* θ*-subgroup. Then an* $\bar{A}$*-signalizer functor* $\bar{\theta}$ *on* $\bar{G}$ *is defined by* $\bar{\theta}(C_{\bar{G}}(\bar{a})) := \overline{\theta(C_G(a))}$, $a \in A^*$, *and*

$$\mathcal{M}_{\bar{\theta}}(\bar{A}) = \{\bar{Y} \mid Y \in \mathcal{M}_\theta(A)\} = \{\bar{K} \mid X \subseteq K \in \mathcal{M}_\theta(A)\}.$$

COMMENTS ON THE ABOVE: 1.1 and 1.2 should be clear and are essentially contained in [4].

On Lemma 1.3: For the proof of $P := [t, O_p(K)] \subseteq O_p(H)$ let $O_p(H) = 1$. Then $C_{F(H)}(t) \subseteq U := C_{F(H)}(P)$, whence t inverts $N_{F(H)}(U)/U$, and hence $[t, P] = P$ centralizes this factor group, consequently centralizing $N_{F(H)}(U)$.

On Proposition 1.4: (i) follows from (1.2.iii) because $O_{\pi'}(K) \cap M$ centralizes U. Apply $(*)$ with $X = Z(O_p(M))$, $p \in \pi$, to get $U_{p'} \subseteq O_{p'}(K)$, and likewise $U_p \subseteq O_{q'}(K)$, $p \neq q \in \pi$, via (1.2.ii). Then $[U_p, O_{p'}(K)] \subseteq O_{\pi'}(K)$, so that $O_{p'}(K) \subseteq O_{\pi'}(K)C_K(U_p)$ and hence $O_{\pi'}(K)U_{p'} \lhd O_{p'}(K)$ which gives (ii). So let $K *** M$. Then $\pi(F(M)) = \pi(F(K))$, by (i). Let $|\pi| \neq 1$. Then (ii) yields $U \subseteq F(K)$, hence $F(K) \subseteq M$, by $(*)$. By symmetry, $F(M) \subseteq K$, hence even $F(M) = F(K)$. Apply $(*)$.

On the Transitivity Theorem: Its proof requires only (1.5), not the less obvious Lemma 2.7 in [2].

On (1.6): This every useful observation of Glauberman is immediate from (1.1.iii). It has replaced a less elementary tool in an earlier version of this note.

Finally, (1.7) is a straightforward consequence of $C_{\bar{G}}(\bar{a}) = \overline{C_G(a)}$ and (1.5). See also Lemma 2.6 and 5.1 in [2].

2. The minimal counterexample G

We begin with our proof of Goldschmidt's theorem, proceeding by contradiction and induction on $|G| + |\pi(\theta)|$. By minimality of G,

2.1 *any proper subgroup* $H \supseteq A$ *of* G *has a unique maximal* θ*-subgroup* $\theta(H)$, *equal to* $\langle H \cap K \mid K \in \mathcal{M}_\theta(A)\rangle$.

We want to apply this to normalizers of non-identity θ-subgroups, and therefore need that

2.2. $1 \neq X \in \mathcal{H}_\theta(A)$ *implies* $N_G(X) \subset G$.

Proof. Suppose $X \lhd G$. Then, in the notation of (1.7), we have $|\mathcal{H}_\theta^*(A)| = |\mathcal{H}_{\bar\theta}^*(\bar A)|$, and, by minimality of G, $|\mathcal{H}_{\bar\theta}^*(\bar A)| = 1$.

2.3. *For* $M \in \mathcal{H}_\theta^*(A)$ *and* $K \in \mathcal{H}_\theta(A)$ *we have* $M * K$. *More explicitly,* $1 \neq X = X^A \lhd M$ *implies* $N_K(X) \subseteq \theta(N_G(X)) = M$.

This is immediate from the above, and is our main starting point. Next we consider the set A_0 of involutions $t \in A$ such that $\theta(C_G(t)) \notin \mathcal{H}_\theta^*(A)$. This means that some $\theta(C_G(t))$-invariant θ-subgroup U_t is not centralized by t. Choose U_t as small as possible, and let

$$\theta(N_G(U_t)) \subseteq M_t \in \mathcal{H}_\theta^*(A) \text{ and } F_t := F(M_t).$$

Lemma 2.4. *Let* $t \in A_0$. (i) U_t *equals* $[t, U_t]$, *is nilpotent, and is centralized by* $C_{F_t}(t)$;

(ii) U_t *lies in* F_t, *and for any* $H \in \mathcal{H}_\theta(A)$ *we have* $[t, H \cap F_t] \subseteq F(H)$.

Proof. Minimal choice of U_t yields $U_t = [t, U_t] = F(U_t) = N_{U_t}(C_{F_t}(t))$, see (1.1), and $[t, U_t] = U_t$ then centralizes $C_{F_t}(t)$.

As U_t is a nilpotent subgroup of M_t invariant under $\theta(C_G(t)) = C_{M_t}(t)$, and since $C_H(t) \subseteq \theta(C_G(t)) \subseteq M_t$, (ii) is immediate from Lemma 1.3.

Lemma 2.5. *Let* $t \in A_0$, $1 \neq X = X^A \subseteq C_{F_t}(t)$, *and* $\theta(N_G(X)) \subseteq H \in \mathcal{H}_\theta^*(A)$. *Then* $H = M_t$.

Proof. By choice of H, $M_t ** H$ and $M_t * H$, see (2.3). By Lemma 2.4, $U_t \subseteq F(H)$, so that $N_H(U_t) \subseteq \theta(N_G(U_t)) \subseteq M_t$ gives $H *** M_t$.

Now we prove that more generally $M_1 *** M_2 *** M_1$ with $M_i \in \mathcal{H}_\theta^*(A)$ and $\theta(C_G(A)) \subseteq M_1$ implies $M_1 = M_2$.

Proposition 1.4.iii leaves us with the case where both $F(M_i)$ are p-groups. Let $P_i = P_i^A \in \mathrm{Syl}_p(M_i)$ and $P_i \subseteq Q_i \in \mathcal{H}_\theta^*(A, p)$.

By the ZJ-Theorem, $1 \neq X_i := Z(J(P_i)) \lhd M_i$, whence $N_K(X_i) \subseteq M_i$ for any $K \in \mathcal{H}_\theta(A)$, see (2.3).

In particular, $N_{Q_i}(X_i) \subseteq P_i$ and hence $P_i = Q_i$.

By the Transitivity Theorem, Q_1 and Q_2 are conjugate under $\theta(C_G(A))$, a subgroup of M_1. Thus $X_1 = X_2$, forcing $M_1 = M_2$.

LEMMA 2.6. *If such an involution $t \in A_0$ exists, then some non-cyclic subgroup B of A centralizes a θ-subgroup $X \neq 1$ such that $X \subseteq F(H)$ whenever $X \subseteq H \in И_\theta(A)$.*

PROOF. (1.2.ii) yields a subgroup B of A with A/B cyclic such that $C_{F_t}(B) \not\subseteq C_{F_t}(t)$. Let $X := [t, C_{F_t}(B)]$ and apply Lemma 2.4.ii.

LEMMA 2.7. *$\theta(C_G(a)) \in И^*_\theta(A)$ for some involution $a \in A$.*

PROOF. Suppose false. Then the above applies to any involution t of A. Let B and X be as in Lemma 2.6. Let $\theta(N_G(X)) \subseteq H \in И^*_\theta(A)$. Then Lemma 2.5 yields $H = M_t$ for each involution $t \in B$, contrary to

REMARK 2.8. *If $H \in И_\theta(A)$, and B is a non-cyclic subgroup of A, then $\theta(C_G(b)) \not\subseteq H$ for some $b \in B$.*

Otherwise, $K = \langle C_K(b) \mid b \in B^* \rangle \subseteq H$ for any $K \in И_\theta(A)$, by (1.1.ii).

3. An application of the Glauberman signalizer functors θ_π

In this section, independent of the assumption that A is a 2-group, we prove

LEMMA 3.1. *$C_A(M) = 1$ for any $M \in И^*_\theta(A)$.*

LEMMA 3.2. *$C_A(P) = 1$ for any $P \in И^*_\theta(A, p)$, $p \in \pi(\theta)$.*

Lemma 3.1 contradicts Lemma 2.7 and is a corollary of Lemma 3.2: With $p \in \pi(F(M))$, $O_p(M) \subseteq P \in И^*_\theta(A, p)$, and $T := C_A(M)$ we have $N_P(O_p(M)) \subseteq M \subseteq C_G(T)$, hence $P = C_P(T)$, see (1.1.i′), as required.

For the proof of Lemma 3.2 suppose $T := C_A(P) \neq 1$.

The Transitivity Theorem yields $\langle И_\theta(A, p) \rangle \subseteq C_G(T)$, whence, for any $H \in И_\theta(A)$, T centralizes $F(H/O_{p'}(H))$ and therefore centralizes $H/O_{p'}(H)$, see (1.2.i) and (1.1.i′). This means $[T, H] \subseteq O_{p'}(H)$.

By minimality of $\pi(\theta)$, the Glauberman signalizer functor $\theta_{p'}$ defined in (1.6) is complete. This defines the θ-subgroup $K := \theta_{p'}(G)$, containing every $\theta(C_G(A))$-invariant p'-subgroup in $И_\theta(A)$. We collect:

$$[T, H] \subseteq O_{p'}(H) \subseteq K \in И_\theta(A) \text{ whenever } \theta(C_G(A)) \subseteq H \in И_\theta(A).$$

This implies $[T, C_K(b)] = [T, \theta(C_G(b)]$ for any $b \in A^*$. In particular, $[T, C_K(b)]$ is invariant under $C_{\theta(C(T))}(b) = C_{\theta(C(b))}(T)$.

Let $T \subseteq B \subseteq A$ with B non-cyclic. The subgroups $[T, C_K(b)]$, $b \in B^*$, are $C_{\theta(C(T))}(B)$-invariant, as we just have seen, and by (1.1.ii′) they generate $[T, K]$.

Thus $C_{\theta(C(T))}(B)$ normalizes $[T, K]$; and since $\theta(C(T))$ is generated by such subgroups (by 1.1.ii and since $r(A) \geqq 3$), it follows that $\theta(C(T))$ normalizes $[T, K]$.

Hence $M := [T, K]\theta(C(T)) \in И_\theta(A)$, see (1.5).

Thus $H = [T, H]C_H(T) \subseteq M$ whenever $\theta(C(A) \subseteq H \in И_\theta(A)$.

In particular, $\theta(C_G(b)) \subseteq M$ for any $b \in A^*$, a contradiction.

References

1. G. Glauberman, *On solvable signalizer functors in finite groups*, (to appear).
2. D. Goldschmidt, *Solvable signalizer functors on finite groups*, J. Algebra **21** (1972), 137–148.
3. D. Goldschmidt, *2-signalizer functors on finite groups*, J. Algebra **21** (1972), 321–340.
4. D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1968.
5. D. Gorenstein, *On the centralizers of involutions in finite groups*, J. Algebra **11** (1969), 243–277.
6. D. Gorenstein, *The flatness of signalizer functors on finite groups*, J. Algebra **13** (1969), 509–512.
7. D. Gorenstein, *On the centralizers on involutions in finite groups II*, J. Algebra **14** (1970), 350–372.
8. D. Gorenstein, *Centralizers of involutions in finite simple groups*, in *Finite Simple Groups*, ed. M. B. Powell and G. Higman, Academic Press, London, 1971.
9. P. Martineau, *Rank 3 signalizer functors on finite groups*, Bull. London Math. Soc. **4** (1972), 161–162.

Mathematisches Seminar der Universität
23 Kiel
Federal Republic of Germany